

An Algorithm of Population Forecast with Fractional Derivative and Variational Iteration Method

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Abstract: An algorithm of population forecast is established by Caputo's fractional derivative, the fractional derivative was extended to modify the Logistic model with harvesting functions, and the variational iteration method is applied to find approximate solutions of the model with Caputo's fractional derivative. As an example of America population forecast, by comparing the Logistic model with harvesting function and the model with fractional derivative, the results of this paper are much closer to the actual situation than that obtained by the classical Logistic model.

Keywords: applied mathematics; Logistic model; Caputo's fractional derivative; Laplace transform; variational iteration method.

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1 Introduction

It goes without saying that prediction of population is of great importance. There are some methods to predict population such as the Logistic model^[1-2], multiple regression model^[3], power-function method^[4], fractional volterra integro-differential method^[5], probabilistic forecasting method^[6], etc.

Due to the growth of population with the traits of memory effect and the locality of ordinary derivative, generally speaking, common models with ordinary derivative (such as the Logistic model) are not very suitable to be used to predict population.

The advantage of fractional-order differential equation is that it allows greater degrees of freedom and incorporates the memory effect in the model^[7]. Utilizing this advantage, we employ in this paper the Caputo's fractional derivative to improve the Logistic model of the following form (LM)

$$\begin{cases} \frac{dx(t)}{dt} = r(t)x(t) \left(1 - \frac{x(t)}{C(t)}\right) - \sigma(t) := f(t, x(t)) \\ x(0) = x_0 \end{cases} \quad (1)$$

as follows (FLM)

$$\begin{cases} {}^C_0D_t^\alpha x(t) = r(t)x(t) \left(1 - \frac{x(t)}{C(t)}\right) - \sigma(t) := \\ f(t, x(t)), \quad 0 < \alpha \leq 1 \\ x(0) = x_0 \end{cases} \quad (2)$$

and the variational iteration method (VIM)^[8] is employed to establish the algorithm of approximate solutions of (2), where ${}^C_0D_t^\alpha$ is the Caputo's derivative of α order, $x(t)$ is the population density at time t , $\sigma(t)$ is the harvesting function, $r(t)$ is the intrinsic growth rate and $C(t)$ is the carrying capacity of environment. As an example of America population prediction, the results of this paper show that the prediction is much closer to the actual situation than that made by the classical Logistic model (see, Table 1).

2 Variational iterative algorithm for solving (2)

Before solving (2) via the variational iterative method (VIM), we recall some notions about the Caputo's derivative and the Laplace transform.

The Caputo's derivative of α order^[9] is defined as

$${}^C_0D_t^\alpha x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{x^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, \quad \alpha > 0, m = [\alpha] + 1, \\ t > 0$$

where $x(t) \in C^m[0, T]$, Γ is gamma function.

The Laplace transform^[9] is defined as

$$\bar{x}(s) = L(x(t)) = \int_0^{\infty} e^{-st} x(t) dt, \quad t > 0,$$

where $\bar{x}(s)$ is a image function, $x(t)$ is the original function of the variables t and a parameter s is complex number.

The inverse Laplace transform is defined as

$$x(t) = L^{-1}(\bar{x}(s)) = \frac{1}{2\pi i} \lim_{T \rightarrow \infty} \int_{p-iT}^{p+iT} e^{st} \bar{x}(s) ds$$

the integrator is carried out in the complex plane along the vertical $\text{Re}(s) = p$ and p satisfies all the singular points on the complex plane are greater than the value of the real part $\bar{x}(s)$.

Lemma 1^[9] Laplace transform of the term ${}_0^C D_t^\alpha x(t)$ holds:

$$L[{}_0^C D_t^\alpha x(t)] = s^\alpha \bar{x}(s) - \sum_{k=0}^{m-1} x^{(k)}(0^+) s^{\alpha-1-k}, \quad m = [\alpha] + 1 \quad (3)$$

Next, we use the variational iterative method (VIM)^[10-11] to solve (2). Generally speaking, applications of VIM for solving differential equations with the initial value conditions follows the three steps: (a) Construction of the correction function; (b) Identification of the Lagrange multiplier; (c) Determination of the initial iteration. Obviously, the second step is crucial. Recently, an easy way to calculate the Lagrange multiplier is suggested in [10-11].

Taking the Laplace transform to both sides of (2) and utilizing Lemma 1 with $m=1$, we have

$$s^\alpha \bar{x}(s) - x(0) s^{\alpha-1} = L[f(t, x(t))] \quad (4)$$

Constructing the correction function of Eq. (4)

$$\bar{x}_{n+1}(s) = \bar{x}_n(s) + \lambda(s) [s^\alpha \bar{x}_n(s) - x(0) s^{\alpha-1} - L[f(t, x_n(t))]]$$

where $n=0, 1, 2, \dots$ and the Lagrange multiplier is taken as $\lambda(s) = -\frac{1}{s^\alpha}$ ^[10-11]. From this we obtain

$$\bar{x}_{n+1}(s) = \frac{x(0)}{s} + \frac{1}{s^\alpha} L[f(t, x_n(t))] \quad (5)$$

Taking the inverse Laplace transform to Eq. (5), the iterative formula for Eq. (2) is obtained as follows

$$x_{n+1}(t) = L^{-1}\left[\frac{x(0)}{s}\right] + L^{-1}\left[\frac{1}{s^\alpha} L[f(t, x_n(t))]\right]$$

that is,

$$x_{n+1}(t) = x(0) + L^{-1}\left[\frac{1}{s^\alpha} L[f(t, x_n(t))]\right] \quad (6)$$

Setting $x_0(t) = x_0$, we obtain variational iterative algorithm $x_n(t)$ of (2).

3 Results of America population forecast and comparison

When $C(t), r(t)$ are constant functions and $\sigma(t) = 0$ for $t \geq 0$, that is, $f(t, x(t)) = rx(t)(1 - \frac{x(t)}{C})$, $C > 0$ and $r > 0$, (1) becomes the classical Logistic model (CLM) and has the exact solution^[12]

$$x(t) = \frac{Cx_0}{x_0 + (C - x_0)e^{-rt}}, \quad 0 < r < 1, C > 0. \quad (7)$$

In general, there is a dependency relationship between $x(t)$ and $\sigma(t)$. In order to illustrate the advantages of (2), we consider the case that $\sigma(t)$ and $x(t)$ are proportional, that is, there exists a constant β such that $\sigma(t) = \beta x(t)$, $t \geq 0$. In this case, (1) becomes a new model (NLM)

$$\begin{cases} \frac{dx(t)}{dt} = rx(t)(1 - \frac{x(t)}{C}) - \beta x(t) \\ x(0) = x_0 \end{cases} \quad (8)$$

and (8) has the exact solution

$$x(t) = -\frac{Cx_0(r - \beta)}{-rx_0 - (rC - rx_0 - \beta C)e^{-(r-\beta)t}} \quad (9)$$

By the iterative formula (6), we have the first order approximation of (8)

$$x_1(t) = x_0 + \frac{kt^\alpha}{\Gamma(\alpha+1)}$$

and the second order approximation of (8)

$$x_2(t) = x_0 + k \left[\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{Cr - C\beta - 2rx_0}{\Gamma(2\alpha+1)C} t^{2\alpha} - \frac{kr\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)C} t^{3\alpha} \right] \quad (10)$$

where $k = rx_0(1 - \frac{x_0}{C}) - \beta x_0$. Utilizing the data from America population websites^[13], the comparison of CLM, NLM, and FLM will be illustrated.

Setting $x_0 = 5.308$ and substituting $x(50) = 23.192$, $x(100) = 76.212$ into (7), we have ($C = 188.12$, $r = 0.031551$)

$$x(t) = \frac{998.546}{5.308 + 182.813e^{-0.031551t}} \quad (11)$$

Setting $x_0 = 5.308$ and substituting $x(50) = 23.192$, $x(100) = 76.212$, $x(150) = 150.697$, into (9), we have ($C = 247.7455279$, $r = 0.04155082143$, $\beta = 0.01$)

$$x(t) = \frac{41.49037962}{0.2205517602 + 7.596023151e^{-0.03155082143t}} \quad (12)$$

The second order approximation(10) of (8) can be obtained for some α ,these results are showed in Figure 1. From the Figure1, we know that (2) with some $\alpha \in [0.7, 0.8]$ has very good agreements with the actual population. Further, the results of $\alpha=0.7, 0.756, 0.8$ are compared in Figure 2.

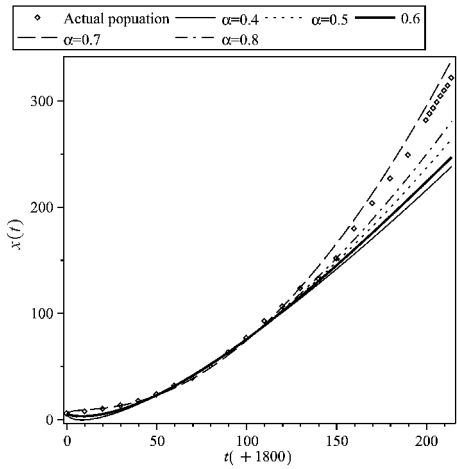


Figure 1 The results with different α

According to the Figure 2,FLM with $\alpha=0.756$ has a good agreement with the actual population compared with $\alpha = 0.7$ and $\alpha = 0.8$. Therefore, setting $\alpha = 0.756$ in (10), we have the second order approximation of (2)

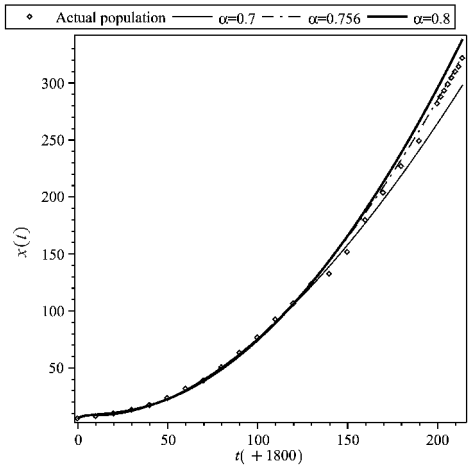
$$x(t)=5.308-0.8915549506t^{0.756}+0.09466986240t^{1.512}-0.0000046993313t^{2.268}, t \in [0, 230]. \tag{13}$$


Figure 2 The optimization of α

Table 1 Comparision of the absolute error between CLM,NLM and FLM

| Year | AP | CLM | AE | NLM | AE | FLM | AE |
|------|---------|---------|---------|---------|---------|---------|--------|
| 1800 | 5.308 | 5.308 | 0 | 5.308 | 0 | 5.308 | 0 |
| 1810 | 7.24 | 7.202 | 0.038 | 7.202 | 0.038 | 3.301 | 3.938 |
| 1820 | 9.638 | 9.735 | 0.097 | 9.735 | 0.097 | 5.497 | 4.141 |
| 1830 | 12.866 | 13.095 | 0.229 | 13.094 | 0.228 | 9.837 | 3.020 |
| 1840 | 17.069 | 17.501 | 0.432 | 17.500 | 0.431 | 15.824 | 1.14 |
| 1850 | 23.192 | 23.192 | 0 | 23.192 | 0 | 23.311 | 0 |
| 1860 | 31.443 | 30.405 | 1.038 | 30.405 | 1.038 | 31.773 | 0.330 |
| 1870 | 39.818 | 39.32 | 0.492 | 39.326 | 0.492 | 41.448 | 1.629 |
| 1880 | 50.189 | 50.034 | 0.155 | 50.034 | 0.155 | 52.124 | 1.935 |
| 1890 | 62.948 | 62.435 | 0.513 | 62.535 | 0.414 | 63.732 | 0.784 |
| 1900 | 76.212 | 76.213 | 0.001 | 76.212 | 0 | 76.212 | 0 |
| 1910 | 92.228 | 90.834 | 1.394 | 90.833 | 1.395 | 89.516 | 2.711 |
| 1920 | 106.022 | 105.612 | 0.41 | 105.712 | 0.31 | 103.603 | 2.419 |
| 1930 | 122.775 | 119.834 | 2.941 | 119.833 | 2.941 | 118.436 | 4.338 |
| 1940 | 132.165 | 132.886 | 0.721 | 132.875 | 0.71 | 133.985 | 1.819 |
| 1950 | 150.697 | 144.354 | 6.972 | 148.354 | 2.343 | 150.221 | 0.475 |
| 1960 | 179.323 | 154.052 | 25.271 | 154.051 | 25.271 | 167.120 | 12.202 |
| 1970 | 203.302 | 161.99 | 41.312 | 161.99 | 41.312 | 184.660 | 18.642 |
| 1980 | 226.546 | 168.316 | 58.23 | 168.316 | 58.23 | 212.819 | 13.726 |
| 1990 | 248.71 | 173.252 | 75.458 | 173.251 | 75.458 | 231.581 | 17.128 |
| 2000 | 281.422 | 177.038 | 104.384 | 177.038 | 104.384 | 260.928 | 20.494 |
| 2010 | 308.746 | 179.905 | 128.841 | 179.906 | 128.840 | 280.843 | 27.902 |
| 2020 | | 182.057 | | 182.057 | | 302.034 | |
| 2030 | | 183.659 | | 183.659 | | 326.459 | |

By setting some values of t , we obtain the numerical results of CLM, NLM and FLM with $\alpha=0.756$ in the table 1 (Unit: million), where AP represents the actual population and AE denotes the absolute error.

When t equals 230, the value of the NLM is close to the carrying capacity of environment. In order to show that the advantages of (2), the t should be constrained in $[0, 230]$. According to table 1, we obtain that the average error of CLM is 42.75, the average error of NLM is 42.07, and that of FLM is 10.14. This shows that FLM has much more advantages to predict the number of population.

4 Conclusions

In this paper, the classical Logistic model is modified by the Caputo's fractional derivatives. The variational iteration formula for the Logistic model with fractional derivatives is suggested. The second approximation solution of the model is given, and the results of comparison between LM, NLM and FLM are illustrated. The results show that FLM has much more advantages to forecast the number of population. It has been well-known that LM has been used in many fields, FLM with Eq. (6) may be applied in these fields in the future.

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基于分数阶导数和变分迭代法的人口预测算法

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摘要:通过对经典的 Logistic 模型进行修正, 构造一种基于分数阶导数的人口预测算法。主要应用分数阶导数对带有收获函数的 Logistic 模型进行修正, 将经典的 Logistic 模型修正为分数阶微分模型, 再用变分迭代法解修正后的 Logistic 模型, 由此可得到分数阶微分模型的各阶近似解。通过预测美国人口比较了带有收获函数的 Logistic 模型和分数阶 Logistic 模型的优缺点。通过比较发现, 分数阶 Logistic 模型能更好的吻合实际数据, 提高预测的精度。

关键词:应用数学; Logistic 模型; Caputo 导数; 拉普拉斯变换; 变分迭代法