

Some Sufficient Conditions for Supersolvability of a Finite Group

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Abstract: Let X be a non-empty subset of G . A subgroup H of a finite group G is said to be X - s -semipermutable in G if H has a supplement T in G such that H is X -permutable with any Sylow subgroup of T for some $x \in X$. Let P be a Sylow p -subgroup of a finite group G , and d a power of p such that $1 \leq d < |P|$. We derive some theorems and corollaries that extend known results concerning S -semipermutable subgroups. We obtained in this paper that if $H \cap O^p(G)$ is X - s -semipermutable in G for all normal subgroups H of G with $|H| = d$, where X is a soluble normal subgroup of G , then either G is p -supersoluble or else $|P \cap O^p(G)| > d$.

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0 Introduction

All groups considered in this paper are finite. Notations and terminologies are standard, and the reader is referred to the texts of W. Guo^[1] and Hupert^[2] for notations and terminologies not introduced. A group G is said to be p -soluble if every chief factor is either a p' -group or a p -group, and G is said to be p -supersoluble if G is p -soluble and every its p -chief factor is of order p . The subgroup $O^p(G)$ of a group G is the subgroup generated by all p' -elements of G , which is also the minimal normal subgroup of G with the quotient is a p group. In the past, many researchers studied the influence of $O^p(G)$ on the structure of G . For example, in^[3], Guo and Issacs obtained the following result:

Theorem A ([3, Theorem B]) Let $P \in \text{Syl}_p(G)$ and d be a power of p such that $1 \leq d < |P|$. Let $U = O^p(G)$ and assume that $H \cap U \triangleleft U$ for all subgroups $H \triangleleft P$ with $|H| = d$. Then either G is p -supersoluble, or else $|P \cap U| > d$.

In the later researches, it is found that the normality of H in Theorem 1.1 in the paper^[4] can be weakened. Let A, B be subgroups of a group G . Then A is said to be permutable with B if $AB = BA$, that is, AB is a subgroup of

G . A subgroup H of a finite group G is said to be permutable (or quasinormal) in G if H permutes with any subgroup of G , and H is said to be S -permutable in G ^[5] if H permutes with any Sylow subgroup of G . If H has a supplement T in G such that H permutes with all Sylow subgroup of T , then H is called S -semipermutable in G . Ball-ester-Bolinches and et al. generalized Theorem 1.1 in the paper^[4] and proved the results.

Theorem B ([5, Theorem 2]) Let $P \in \text{Syl}_p(G)$ and let d be a power of p such that $1 \leq d < |P|$. Let $U = O^p(G)$ and assume that $H \cap O^p(G)$ is S -semipermutable in G for all subgroups $H \triangleleft P$ with $|H| = d$. Then either G is p -supersoluble, or else $|P \cap U| > d$.

Let X be a subset of a group G and H, T be subgroups of G . Then H is said to be X -permutable^[6] with T in G if $HT^x = T^xH$ for some $x \in X$. Clearly, a permutable subgroup of G is a subgroup that 1-permutes with all subgroup of G . And, if $X = G$, a subgroup which X -permutes with all subgroup of G is X -permutable in G ^[7]. X -permutability of subgroups is studied extensively and in^[8], the following definition were proposed.

Definition [8, definition 1] Let A be subgroup of a group G and X a non-empty subset of G . Then A is said to be X - s -semipermutable in G if A has a supplement T in G such that A is X -permutable with all Sylow subgroups of T .

It is easy to find that a S -semipermutable subgroup is X - s -semipermutable subgroup in a group G for any sub-

group X of G , but an X -s-permutable subgroup is not necessarily to be S -permutable in G if X is not 1. We derive some theorems and corollaries that extend known results concerning S -semipermutable subgroups^[9-15]. To develop the work in^[3], in this paper, we obtained the following theorem.

Theorem C Let G be a finite group and X a soluble normal subgroup of G . Let p be a prime. Let $P \in Syl_p(G)$ with p be a prime and let d be a power of p such that $1 \leq d < |P|$. Write $U = O^p(G)$, and assume that $H \cap O^p(G)$ is X -s-semipermutable in G for all subgroups $H \triangleleft P$ with $|H| = d$. Then either G is p -supersoluble, or else $|P \cap U| > d$.

1 Some Lemmas

To prove Theorem, we list some known results as lemmas in this section.

Lemma 1.1 [8, Lemma 2.1] Let A and X be subgroups of G and $N \triangleleft G$. Then:

(1) If H is a permutable subgroup of G , A is X -s-semipermutable in G , then HA is a X -s-semipermutable subgroup of G .

(2) If A is X -s-semipermutable in G and $T \in X_s(A)$, then AN/N is XN/N -s-semipermutable in G/N and $TN/N \in (XN/N)_s(AN/N)$.

(3) If A/N is XN/N -s-semipermutable in G/T and $T/N \in (XN/N)_s(A/N)$, then A is X -s-semipermutable in G and $T \in X_s(A)$.

(4) If A is X -s-semipermutable in G , $A \leq D \triangleleft G$ and $X \leq D$, then A is X -s-semipermutable in D .

(5) If $T \in X_s(A)$ and $A \leq N_G(X)$, then $T^x \in X_s(A)$, for any $x \in G$.

(6) If A is X -s-semipermutable in G and $X \leq D$, then A is D -s-semipermutable in G .

Lemma 1.2 [16, Lemma 3.13] Let A and B be some subgroups of a group G such that $G \neq AB$ and $AB^x = B^xA$ for each $x \in G$. Then G has a proper normal subgroup N such that either $A \leq N$ or $B \leq N$.

Lemma 1.3 [16, Lemma 2.5] Let G be a group, P a p -subgroup of G and Q a q -subgroup of G , where p, q are different primes dividing $|G|$. If L is a subnormal subgroup of G and $PQ = QP$, then $PQ \cap L = (P \cap L)(Q \cap L)$.

2 Proof of Theorem C

Assume that the result is not true and let G be a counterexample of least order. Write $U = O^p(G)$, $K = P \cap U$. Then $|K| \leq d$ and G is not p -supersoluble. In particular, $K \neq 1$ and $d \geq p$. Write $\mathfrak{h} = \{H \triangleleft P \mid |H| = d\}$. By hypothesis, $H \cap U$ is X -s-semipermutable in G for each $H \in \mathfrak{h}$.

Step 1 $O_p(G) = 1$

Write $V = O_{p'}(G)$ and assume that $V \neq 1$. Consider the factor group $\bar{G} = G/V$. Let \bar{H} be a normal subgroup of \bar{P} of order d . Then there is $H \in \mathfrak{h}$ such that $\bar{H} = HV/V$. Since $H \cap U$ is X -s-semipermutable in G , we have $\bar{H} \cap \bar{U} = (H \cap U)V/V$ is X -s-semipermutable. Since $|\bar{U} \cap \bar{P}| \leq d$, it follows that \bar{G} is p -supersoluble and this is a contradiction. Thus, $V = 1$.

Step 2 K is X -s-semipermutable in G .

Since K is normal in P of order at most d , there exists $H \in \mathfrak{h}$ such that $K \leq H \leq P$. Then $K = P \cap U$ is X -s-semipermutable in G .

Step 3 Let M be a maximal subgroup of P . Then $M \cap U = M \cap K$ is X -s-semipermutable in G .

Since $|K| \leq d$ and $M \cap U \triangleleft P$, there exists an $H \in \mathfrak{h}$ such that $M \cap K \leq M \cap U \leq H \leq M$. Then $M \cap K = M \cap K \cap H = H \cap K = H \cap P \cap U = H \cap U$. Thus, $M \cap K = M \cap U = H \cap U$ is X -s-semipermutable in G .

Step 4 $O_p(G) \cap U \neq 1$

Suppose that $O_p(G) \cap U = 1$ and let R be a minimal normal subgroup of G contained in U . Then R is not abelian. Since R is neither a p -group nor a p' -group by step 1. So, $R \cap X = 1$ and hence $R \subseteq C_G(X)$. Let L be a minimal normal subgroup of R . Then L is a non-abelian simple subgroup. Let q be a prime divides order of L different from p and Q a Sylow q -subgroup of L .

We claim that $K \cap L$ permutes with Q . In fact, by step 2, K is X -s-semipermutable in G .

Let $T \in X_s(K)$. Then the Sylow q -subgroup T_q is also a Sylow q -subgroup of G , and $KT_q^x = T_q^x K$. On the other hand, Q is contained in some conjugate of T_q . By Lemma 1.1 (5), without loss of generality, we may assume $Q \subseteq T_q$. Since $L \subseteq R \subseteq C_G(X)$, it follows that $Q^x = Q = T_q \cap L$ is a Sylow q -subgroup of L . By lemma 1.3 $L \cap KT_q^x = (L \cap K)(L \cap T_q^x) = (L \cap K)Q$ is a subgroup of L , that is, $L \cap K$ permutes with Q and our claim holds.

Since Q is an arbitrary Sylow q -subgroup of L , and any Sylow q -subgroup of L is conjugate of Q , we have that $L \cap K$ permutes with any conjugate of Q in L , that is $(L \cap K)Q^a = Q^a(L \cap K)$ for any $a \in L$. Clearly $(L \cap K)Q \neq L$, so L is not simple by lemma 1.2, a contradiction. This contradiction shows that $U \cap O_p(G) \neq 1$ and step 4 holds.

Step 5 $d > p$

Assume that $d=p$. Then K is of order 1 or p . By step 4, U is p -soluble, hence $P \cap U$ is contained in p -supersoluble hypercentre. Therefore G is p -supersoluble, a contradiction.

Step 6 Final contradiction.

Let N be a minimal normal subgroup of G contained in U . Since $|K| \leq d$. We have $|N| \leq d$. Suppose that $|N| < d$. We argue that G/N satisfies the hypothesis of the theorem. Clearly, $1 \leq d/|N| < |P/N|$. Let H/T be a normal subgroup of P/T of order $d/|T|$. Then $H \in \mathfrak{h}$. It follows that $H/N \cap O^p(G/N) = H/N \cap U/N = (H \cap U)/N$, which is X -s-semipermutable in G/N . This shows that G/N satisfies the hypothesis of the claimed. Since $|P/N \cap U/N| = |(P \cap U)/N| \leq d/|N|$, it follows that G/N is p -supersoluble by minimality of G . Let $|N| = d$. Then $N = K$. So G/N is p -supersoluble.

By [7, Kapitel VI, Satz 8.6], we may suppose that $N \not\subseteq \Phi(P)$. Let M be a maximal subgroup of P such that $N \not\subseteq M$. It is easy to see that $M \cap K$ is a maximal subgroup of K and $M \cap K \neq 1$. By step 3, $M \cap K$ is X -s-semipermutable in G . Take a subgroup $T \in X_s(M \cap K)$. Then $G = (M \cap K)T$ and for any Sylow q -subgroup Q of T , where $q \neq p$, we have $(M \cap K)Q^x = Q^x(M \cap K)$ for some $x \in X$. Thus, $N \cap M = N \cap (M \cap K)Q^x \triangleleft (M \cap K)Q^x$ and so $N \cap M$ is normalized by Q^x . Clearly $N \cap M$ is a normal subgroup of P , so we have that $N \cap M \triangleleft G$. This implies that $N \cap M = 1$, and so N is of order p . Hence G is p -supersoluble, which is a contradiction. This final contradiction completes the proof.

3 Some Corollaries of Theorem C

Clearly, Theorems A and B can be obtained by Theorem C by choose $X=1$. By Theorem C, we can also obtain some new descriptions of groups. For example, we have

Corollary Let $P \in \text{Syl}_p(G)$ with $|P| > p$ and X a soluble normal subgroup of G . Suppose that, for every

maximal subgroup H of P , $H \cap O^p(G)$ is X -s-semipermutable in G . Then G is p -supersoluble.

Proof With a contradiction in mind, assume that G is not p -supersoluble. Write $U = O^p(G)$. By Theorem C, $P \cap U = P$, that is, $P \leq U$, and so $G = U$. This means that every maximal subgroup of P is X -s-semipermutable in G . Then we conclude that G is p -supersoluble (cf. [14, Theorem 1]).

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有限群超可解的一些充分条件

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摘要: G 是有限群且 X 是一个非空集合. 若子群 H 在 G 中有补充 T , 且对任取 X 中的元 x , H 与 T 的任意 Sylow 子群是 X -置换的, 子群 H 被称为是在 G 中 X -s-半置换的. 令 d 是一个小于 P 的阶的 p -子群的阶. 推广了 S -半置换子群的一些结果, 利用 X -s-半置换子群的性质进一步研究有限群, 给出有限群超可解的一些结论. 即可得到: 对任意的 d 阶正规子群 H 和 G 的可解正规子群 X , 若 $H \cap O^p(G)$ 在 G 中 X -s-半置换的, 则 G 是 p -超可解的或者是 $|P \cap O^p(G)| > d$.

关键词: 基础数学; 代数学; 有限群; p -超可解; X -s-半置换; p -群