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Some Results on Chatterjea Type Nonexpansive Mapping with Additional Conditions

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Abstract: In this paper, we compare Reich type nonexpansive mapping, Chatterjea type nonexpansive mapping and (c)-mapping, and draw some conclusions. Under the premise of $0<c\leqslant b$, we prove that the Chatterjea type nonexpansive mapping T is asymptotically regular and in a UCED Banach space, the Chatterjea type nonexpansive mapping T has a fixed point. Finally, we also find that the Chatterjea type nonexpansive mapping T is orbitally continuous and k-continuous on orbits.

Keywords: Chatterjea type; asymptotically regular, ; fixed point; nonexpansive mapping

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0 Introduction

We know that every nonexpansive mapping or asymptotically nonexpansive mapping on a non-empty closed, bounded, convex subset of a uniformly convex Banach space has at least one fixed point, see ref. [1-3].

In ref. [4], it shows that if Reich type nonexpansive and Chatterjea type nonexpansive mappings satisfy some additional mild conditions, then there exist an AFPS for the mappings. Further, it shows that the respective approximate fixed point sequences converge strongly to a fixed point of the respective mappings. In this paper, we talk about properties of the Chatterjea type nonexpansive mappings satisfy some additional conditions. For more considerations on Reich contractions and Chatterjea contractions see ref. [5-6], respectively ref. [7]. For Hardy-Rogers contractions see ref. [8].

In ref. [9], it solves two fixed point problems associated with the class of (c)-mapping. In this paper, we extend (c)-mapping to the Chatterjea type nonexpansive mapping and also prove that the Chatterjea type nonexpansive mapping is asymptotically regular.

1 Notations and preliminaries

Definition 1[4] Let $T:X\rightarrow X$ be a mapping acting on a metric space (X,d) and let $(x_n)_n$ be a sequence in X. $(x_n)_n$ is said to be an approximate fixed point sequence (in short, AFPS) for T if:

$$\lim_{n\to\infty} d(x_n - T(x_n)) = 0$$

Many authors have studied a number of methods for iterative AFPS, see ref. [10-13].

Definition 2[4] Let X be a normed linear space, C a non-empty subset of X and $T:C\rightarrow C$ be a mapping. The mapping T is said to be a Chatterjea type nonexpansive mapping if there exists non-negative real numbers a,b,c with $a+b+c=1$, such that the condition

$$\| Tx-Ty \| \leqslant a\| x-y \| +b\| x-Ty \| +c\| y-Tx \| \tag{1}$$

holds for all $x,y\in C$. We say that T is a Chatterjea type nonexpansive mapping with coefficients (a,b,c).

Definition 3[9] A Banach space X is said to be uniformly convex in every direction (in short, UCED) if for all $\varepsilon\in(0,2]$ and all $x\in S_X$, there exists $\delta(\varepsilon,x)>0$ such that:

$$\| x_1+x_2 \| \leqslant 2(1-\delta(\varepsilon,x))$$

for all $x_1,x_2\in\overline{B_X}$ and $x_1-x_2\in\{tx, t\in[-2,-\varepsilon]\cup[\varepsilon,2]\}$.

Definition 4[9] A Banach space X is said to be uniformly convex if X is UCED and

$$\inf\{\delta(\varepsilon,x), x\in S_x\}>0$$

for all $\varepsilon\in(0,2]$.

Remark 1 We have the following implications:

Uniform convexity⇒UCED⇒Strict convexity.

Lemma 1[14] A Banach space X is UCED if and

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only if for every bounded sequence $(x_n)_n$ in X, the function f defined on X by $f(x)=\overline{\lim\limits_{n\to\infty}}\|x_n-x\|$ is strictly quasiconvex, that is:

$$f(\alpha y_1+(1-\alpha)y_2)<\max\{f(y_1),f(y_2)\}$$

for all $\alpha\in(0,1)$ and all $y_1,y_2\in X$ with $y_1\neq y_2$.

Definition 5[15] If T is a self mapping of a metric space (X,d) then the set $O(T,x)=\{T^n x:n=0,1,2,\cdots\}$ is called the orbit of T at x and T is called orbitally continuous at a point $z\in X$ if for any sequence $x_n\subset O(T,x)$ for some $x\in X$, $x_n\to z$ implies $Tx_n\to Tz$ as $n\to\infty$.

Definition 6[16] A self-mapping T of a metric space (X,d) is called k-continuous, $k=1,2,\cdots$, if $T^k x_n\to Tz$ whenever $\{x_n\}$ is a sequence in X such that $T^{k-1}x_n\to z$ as $n\to\infty$.

2 Main results

Lemma 2 Let (X,d) be a bounded metric space and let $T:X\to X$ be a Chatterjea type nonexpansive mapping with $0<c\leqslant b$. Then, T is asymptotically regular, that is:

$$\lim_{n\to\infty}d(T^{n+1}x,T^n x)=0$$

for all $x\in X$.

Proof. For $x_0\in C$, let $x_n=T^n x_0$. Then, for $n\geqslant 1$, by (1), we get

$$d(x_{n+1},x_n)=d(Tx_n,Tx_{n-1})$$
$$\leqslant ad(x_n,x_{n-1})+bd(x_n,Tx_{n-1})+cd(x_{n-1},Tx_n)$$
$$\leqslant ad(x_n,x_{n-1})+c(d(x_{n-1},x_n)+d(x_n,Tx_n))$$
$$\leqslant(a+c)d(x_n,x_{n-1})+cd(x_n,x_{n+1})$$
$$\Rightarrow d(x_{n+1},x_n)\leqslant\frac{a+c}{1-c}d(x_n,x_{n-1})=\frac{1-b}{1-c}d(x_n,x_{n-1})\leqslant d(x_n,x_{n-1})$$

Therefore, the sequence $\{d(x_{n+1},x_n)\}$ is nonincreasing, so that $\lim\limits_{n\to\infty}d(x_{n+1},x_n)=r$ exists. We must show that $r=0$. Suppose $r>0$. Then there exists a positive integer s such that the diameter of $c=d_1<\frac{(s+1)r}{2}$. Since $c>0$, there exists $\varepsilon>0$ such that $\{1-(s+1)c^s\}(r+\varepsilon)+\frac{(s+1)rc^s}{2}<r$(this is possible for $0<\varepsilon\leqslant\frac{(s+1)rc^s}{2}$). Then there exists a positive integer N such that $n\geqslant N$ implies $r\leqslant d(x_{n+1},x_n)<r+\varepsilon$.

Now we claim that, for $n\geqslant 0$ and $k\geqslant 1$,

$$d(x_{n+k+1},x_{n+k})\leqslant\{1-(k+1)c^k\}d(x_{n+1},x_n)+c^k d(x_{n+k+1},x_n) \tag{2}$$

To prove (2), let $k=1$. Then, we get, by (1),

$$d(x_{n+2},x_{n+1})\leqslant ad(x_{n+1},x_n)+cd(x_{n+2},x_n)$$
$$\leqslant(1-2c)d(x_{n+1},x_n)+cd(x_{n+2},x_n),$$

which asserts (2) since $b\geqslant c>0\Rightarrow a\leqslant 1-2c$. In order to use induction for k, assume that (2) is true for $k\geqslant 1$. Since $a+b+c=1$, $0<c\leqslant b$, then we have

$$d(x_{n+k+2},x_{n+k+1})\leqslant[1-(k+1)c^k]d(x_{n+2},x_{n+1})+c^k d(x_{n+k+2},x_{n+1})$$
$$\leqslant[1-(k+1)c^k]d(x_{n+1},x_n)+c^k[a(k+1)+bk]d(x_{n+1},x_n)+c^{k+1}d(x_{n+k+2},x_n)$$
$$\leqslant[1-(2+k)c^{k+1}]d(x_{n+1},x_n)+c^{k+1}d(x_{n+k+2},x_n)$$

by using $d(x_{n+k+1},x_n)\leqslant(k+1)d(x_{n+1},x_n)$ and $d(x_{n+k+1},x_{n+1})\leqslant kd(x_{n+1},x_n)$,

which proves (2).

Then, for $n\geqslant N$ and $k=s$, by (2), We have

$$d(x_{n+s+1},x_{n+s})\leqslant\{1-(s+1)c^s\}(r+\varepsilon)+c^k d_1$$
$$\leqslant\{1-(s+1)c^s\}(r+\varepsilon)+\frac{c^k(s+1)r}{2}$$
$$<r$$

which leads a contradiction. Therefore, we have $r=0$.

Remark 2 Let $T:X\to X$ be a mapping acting on a metric space (X,d). If T is asymptotically regular, then for every $x_0\in X$, the sequence $(T^n(x_0))_n$ is an approximate fixed point sequence for T.

Theorem 1 Let X be a UCED Banach space and let $K=\bigcup\limits_{i=1}^{\infty}K_i$ be a finite union of nonempty weakly compact convex subsets K_i, $(i=1,2,\cdots,n)$ of X. Assume that T: $K\to K$ is a Chatterjea type nonexpansive mapping with $x_i\in K_i$, $Tx_i\in K_i$ and $0<c\leqslant b$. Then, T has a fixed point in K.

Proof. Let z_0 be an arbitrary point in K. Define the convex functional $g:X\to[0,+\infty)$ by:

$$g(x)=\overline{\lim_i}\|x-T^i z_0\|$$

Since K is bounded and following Lemma 1, g is strictly quasiconvex. On the other hand, for $1\leqslant i\leqslant n$, K_i is weakly compact by hypothesis. The fact that g is weakly lower semi-continuous implies the existence of $x_i\in K_i$, such that:

$$g(x_i)=\min\{g(x):x\in K_i\}$$

Since T: $K\to K$ is a Chatterjea type nonexpansive mapping, there exist $a,b,c\in[0,1]$, $b\geqslant c>0$ such that $a+b+c=1$ and:

$$\|Ty_1-Ty_2\|\leqslant a\leqslant y_1-y_2\|+b\|Ty_2-y_1\|+c\|Ty_1-y_2\|$$

for all $y_1, y_2 \in K$.

This gives that:

$\| Tx_k - T^i z_0 \| \leqslant a \| x_k - T^{i-1} z_0 \| + b \| T^i z_0 - x_k \| + c \| Tx_k - T^{i-1} z_0 \|$

By the triangle inequality, we obtain that for $1 \leqslant k \leqslant n$:

$\| Tx_k - T^i z_0 \| \leqslant a(\| x_k - T^i z_0 \| + \| T^i z_0 - T^{i-1} z_0 \|) + b \| x_k - T^i z_0 \| + c(\| Tx_k - T^i z_0 \| + \| T^i z_0 - T^{i-1} z_0 \|)$

hence

$(1-c) \| Tx_k - T^i z_0 \| \leqslant (a+b) \| x_k - T^i z_0 \| + (a+c) \| T^i z_0 - T^{i-1} z_0 \|$

Similarly:

$$\| Tx_k - T^i z_0 \| \leqslant \frac{a+b}{1-c} \| x_k - T^i z_0 \| + \frac{a+c}{1-c} \| T^i z_0 - T^{i-1} z_0 \|$$

Since, $a+b+c=1$, $c \leqslant b$ it follows that $\frac{a+b}{1-c}=1$, $\frac{a+c}{1-c}=\frac{1-b}{1-c} \leqslant 1$, which gives that:

$$\overline{\lim_i} \| Tx_k - T^i z_0 \| \leqslant \overline{\lim_i} \| x_k - T^i z_0 \| + \overline{\lim_i} \| T^i z_0 - T^{i-1} z_0 \|$$

Furthermore, since K is bounded, Lemma 2 shows that T is asymptotically regular, which leads to:

$$\overline{\lim_i} \| T^i z_0 - T^{i-1} z_0 \| = \overline{\lim_i} \| T^i z_0 - T^{i-1} z_0 \| = 0$$

It follows that:

$$\overline{\lim_i} \| Tx_k - T^i z_0 \| \leqslant \overline{\lim_i} \| x_k - T^i z_0 \|$$

this proves that:

$$g(Tx_k) \leqslant g(x_k), 1 \leqslant k \leqslant n$$

If there exists $1 \leqslant k_0 \leqslant n$, such that $Tx_{k_0} \in K_{k_0}$ and since $g(x_{k_0})$ is the minimum of g on K_{k_0}, we get $Tx_{k_0} = x_{k_0}$. Indeed, if $Tx_{k_0} \neq x_{k_0}$ and since g is strictly quasiconvex, we obtain:

$$g(x_{k_0}) \leqslant g\left(\frac{x_{k_0}+Tx_{k_0}}{2}\right) < \max\{ g(x_{k_0}), g(Tx_{k_0}) \} = g(x_{k_0})$$

which is a contradiction. x_{k_0} is a fixed point for T.

Theorem 2 Let X be a Banach space and C be a non-empty closed, convex, bounded subset of X. Let $T: C \to C$ be a Chatterjea type nonexpansive mapping with coefficients (a, b, c), such that $b, c<1$. Also assume that for $x, y \in C$

$$\frac{1-b}{7} \| x-Ty \| \leqslant \| x-y \| \Rightarrow \| Tx-Ty \| \leqslant \| x-y \|$$

Further, assume that for any $\varepsilon>0$, there exists $\delta>0$ such that

$$\| x-y \| + \| x-Ty \| + \| y-Tx \| < 3\varepsilon+\delta \Rightarrow \| Tx-Ty \| \leqslant \frac{\varepsilon}{2}$$

Then, T is orbitally continuous and k-continuous.

Proof. In Theorem3.5[1], we know that (x_n) is an AFPS of T. In Theorem 3.6[1], (x_n) is a Cauchy sequence and hence convergent to some $z \in C$.

We obtain that,

$\| Tx_n - Tx_m \| \leqslant a \| x_n - x_m \| + b \| x_n - Tx_m \| + c \| x_m - Tx_n \|$
$\leqslant a \| x_n - x_n \| + b \| x_n - x_m \| + b \| x_m - Tx_m \| + c \| x_n - x_m \| + c \| x_n - Tx_n \|$
$\leqslant \| x_n - x_m \| + b \| x_m - Tx_m \| + c \| x_n - Tx_n \| \to 0$ *as* $n, m \to \infty$

Therefore, (Tx_n) is a Cauchy sequence in C. Also, since $x_{n+1} = \frac{1}{2}(Tx_n + x_n)$, we have that $Tx_n = 2x_{n+1} - x_n \to z$ as $n \to \infty$.

From Theorem 3.6[1], we know that z is a fixed point of T, i.e., $z = Tz$. Then, $Tx_n \to z = Tz$ as $n \to \infty$. Therefore, T is orbitally continuous.

Since $T^{k-1} x_n \to z$, k-continuity of T implies that $T^k x_n \to Tz$. So, we will prove that $T^k x_n \to Tz$.

Again

$\| T^k x_n - Tz \| \leqslant a \| T^{k-1} x_n - z \| + b \| T^{k-1} x_n - Tz \| + c \| z - T^k x_n \|$
$\leqslant a \| T^{k-1} x_n - z \| + b \| T^{k-1} x_n - z \| + b \| z - Tz \| + c \| z - Tz \| + c \| Tz - T^k x_n \|$,
$\Rightarrow (1-c) \| T^k x_n - Tz \| \leqslant \| (b+c) \| z - Tz \|$

Then

$$\| T^k x_n - Tz \| \leqslant \frac{b+c}{1-c} \| z - Tz \| \to 0 \text{ as } n \to \infty$$

Therefore, T is k-continuous.

References:

[1] Browder F E. Nonexpansive nonlinear operators in a Banach space[J]. Proc. Natl. Acad. Sci. USA, 1965, 54: 1041-1044.

[2] Goebel K, Kirk W A. A fixed point theorem for asymptotically nonexpansive mappings [J]. Proc. Am. Math. Soc, 1972, 35: 171-174.

[3] Tomizawa Y. Asymptotically quasi-nonexpansive mappings with respect to the Bregman distance in the intermediate sense[J]. Fixed Point Theory, 2017, 18: 391-406.

[4] Sumit Som, Adrian Petru, Hiranmoy Garai, et al. Some characterizations of Reich and Chatterjea type nonexpansive mappings[J]. Fixed Point Theory Appl, 2019.

[5] Reich,S. Fixed point of contractive functions[J]. Boll. Un. Mat. Ital,1972,5:26-42.
[6] Reich S. Some remarks concerning contraction mappings[J]. Canad. Math. Bull,1971,14:121-124.
[7] Chatterjee S K. Fixed point theorems. C. R[J]. Acad. Bulgare Sci,1972,25:727-730.
[8] Hardy G E,Rogers T D. A generalization of a fixed point theorem of Reich[J]. Canad. Math. Bull, 1973,16:201-206.
[9] Sami Atailia, Najeh Redjel, Abdelkader Dehici. Some fixed point results for (c)-mappings in Banach spaces[J]. Fixed PointTheory Appl,2020.
[10] Bruck R,Kuczumow T,Reich S. Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property [J]. Colloquium Math,1993,65:169-179.
[11] Chidume C E. Global iteration schemes for strongly pseudo-contractive maps[J]. Proc. Am. Math. Soc,1971,126(9):2641-2649.
[12] Rhoades B E. Fixed point iterations for certain nonlinear mappings[J]. Math. Anal. Appl,1994, 183:118-120.
[13] Schu J. Iterative construction of fifixed points of asymptotically nonexpansive mappings[J]. Math. Anal. Appl,1991,158:407-413.
[14] Suzuki T. Fixed point theorems and convergence theorems for some generalized nonexpansive mappings[J]. Math. Anal. Appl,2008,340:1088-1095.
[15] Ciri′c L. On contraction type mappings [J]. Math. Balk,1971,1:52-57.
[16] Pant A,Pant R P. Fixed points and continuity of contractive maps [J]. Filomat, 2017, 31 (11): 3501-3506.

关于带有附加条件的 Chatterjea 类型非扩张映射的一些结果

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摘要:比较 Reich 型非扩张映射、Chatterjea 型非扩张映射和(c)映射,得到一些结论。在 $0<c\leqslant b$ 这个假设的前提下,证明了 Chatterjea 型非扩张映射是渐近正则的,并且在 UCED Banach 空间中,Chatterjea 型非扩张映射具有一个不动点。最后,证明了 Chatterjea 型非扩张映射在轨道上是连续的也是 k-连续的。

关　键　词:Chatterjea 型;渐进正则;不动点;非扩张映射